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1991 J. Phys. A: Math. Gen. 24 L1277

(http://iopscience.iop.org/0305-4470/24/21/005)

J. Phys. A: Math. Gen. 24 (1991) L1277–L1283. Printed in the UK

LETTER TO THE EDITOR

On a general framework for q-particles, paraparticles and q-paraparticles through deformations

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Received 29 July 1991

Abstract. We propose a unification of deformations associated with the descriptions of q-, para- and q-para-bosons and -fermions for complex values of q.

Canonical Bose- and Fermi-quantizations, q-Bose- and q-Fermi-quantizations, para-Bose- and para-Fermi-quantizations as well as q-para-Bose- and q-para-Fermi-quantizations do evidently deal with creation and annihilation operators associated with Bose-like and Fermi-like oscillators respectively.

By limiting (for simplicity) our considerations to one-dimensional contexts, we propose here *to unify* all the quantum deformations of Lie (super)algebras and Lie(super)groups which have been recently suggested for the description of bosons and fermions, q-bosons and q-fermions, parabosons and parafermions as well as q-parabosons and q-parafermions. This general framework corresponds to the following typical set of relations (1), (2) and (3):

$$cc^{\dagger}+g(q)c^{\dagger}c=h(q)=|[N+1]|+g(q)|[N]| \tag{1}$$

$$c|n\rangle=\sqrt{|[n]|}\,|n-1\rangle \tag{2}$$

$$c^{\dagger}|n\rangle=\sqrt{|[n+1]|}\,|n+1\rangle. \tag{3}$$

Here $g(q)$ is an ordinary function while $h(q)$ stands in general for an operator, both symbols having to be discussed in the following. Moreover the action of our annihilation (c) and creation ($c^{\dagger}$) oscillator-like operators on normalized state vectors is readily expressed in terms of *absolute* values of brackets of numbers (n and $n+1$) to be hereafter defined and discussed.

Let us mention at the start that our unification set of relations exploits a set of *similar* equations which has recently been pointed out [1] for only q-bosonic and q-fermionic harmonic oscillators superposed in order to construct a q-analogue of the *supersymmetric oscillator* (see, for example, equations (3), (8) and (11) in [1] with the *ad hoc* definitions). Moreover we want to mention another recent approach [2] for also presenting a general framework accommodating various methods of quantization so far proposed for harmonic oscillators. This will be compared with ours at the end of this letter.

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0305-4470/91/211277+07$03.50

Just before the discussion of the set (1)–(3) within a unifying point of view, let us extract some results needed for evaluating its interest. We notice that (1) can be iterated without difficulty in order to get the information

$$c(c^\dagger)^n+(-1)^{n-1}g^n(q)(c^\dagger)^n c=\sum_{k=0}^{n-1}(-1)^k g^k(q)(c^\dagger)^k h(q)(c^\dagger)^{n-k-1}. \tag{4}$$

This helps us for the complete determination of a normalized basis $\{|n\rangle\}$ where the states $|n\rangle$ are such that

$$|n\rangle=c_n(c^\dagger)^n|0\rangle \qquad c|0\rangle=0 \tag{5}$$

when, evidently, we define $|0\rangle$ as the *vacuum* state ($c_0=1$). The nth eigenstate refers to the n-eigenvalue of the number operator N defined in this oscillator-like context by

$$[c,N]=c \qquad [c^\dagger,N]=-c^\dagger \qquad N^\dagger=N. \tag{6}$$

The whole set of normalization coefficients can be determined, if we characterize our operator $h(q)$ as admitting the vectors $|n\rangle$ as eigenvectors. We then define

$$h(q)|n\rangle=\alpha_{n+1}|n\rangle \tag{7}$$

where we have introduced the notation α_{n+1} for simplifying the coming developments. With the properties (4) and (7), we learn that

$$\langle 0|c^n(c^\dagger)^n|0\rangle=\sum_{k=0}^{n-1}(-1)^k g^k(q)\alpha_{n-k}\langle 0|c^{n-1}(c^\dagger)^{n-1}|0\rangle \tag{8}$$

and consequently that

$$\langle 0|c^n(c^\dagger)^n|0\rangle=\prod_{k=1}^{n}\left(\sum_{j=0}^{k-1}(-1)^j g^j(q)\alpha_{k-j}\right). \tag{9}$$

The normalization coefficients finally appear as given by

$$|c_n|^2=\left\{\prod_{k=1}^{n}\left(\sum_{j=0}^{k-1}(-1)^j g^j(q)\alpha_{k-j}\right)\right\}^{-1}. \tag{10}$$

Our operators c and $c^\dagger$ in (1)–(3) have to refer either to Bose-like *or* to Fermi-like considerations: we thus correspondingly adopt the notations $(b,b^\dagger)$ *or* $(f,f^\dagger)$ for self-consistency within these developments.

Let us analyse the *Bose-like context* as a first step and let us consider successively (i) q-bosons, (ii) parabosons and (iii) q-parabosons in connection with the set of equations (1)–(3).

(i) *For q-bosons* we can in particular pick out the information [1] concerning the coefficients and the bracket $[n]$, i.e.

$$|c_n|^2=\frac{1}{|[n]!|} \qquad [n]=\frac{q^{1/2}}{1-q}(q^{-n/2}-q^{n/2}) \tag{11}$$

so that, from our relation (10), we get

$$|[n]!|=\prod_{k=1}^{n}\left(\sum_{j=0}^{k-1}(-1)^j g^j(q)\alpha_{k-j}\right). \tag{12}$$

This leads to the eigenvalues of the operator $h(q)$ on the form

$$\alpha_{n+1}=|[n+1]|+g(q)|[n]|. \tag{13}$$

The corresponding q-bosonic deformation is thus characterized by

$$bb^\dagger + g(q)b^\dagger b = |[N+1]| + g(q)|[N]| = h(q) \tag{14}$$

and leads to [1, 3]

$$h(q) = q^{-N/2} \tag{15}$$

if $g(q) = -q^{1/2}$. We thus get (1) as a typical commutation relation reducing to the expected commutator when $q \to 1$ in the bosonic case.

(ii) *For parabosons* [4, 5], let us remember that it is necessary to distinguish between even and odd characteristics of the states in the sense that we have

$$b|2n\rangle = \sqrt{2n}|2n-1\rangle \qquad b^\dagger|2n\rangle = \sqrt{2n+p}|2n+1\rangle \tag{16}$$

and

$$b|2n+1\rangle = \sqrt{2n+p}|2n\rangle \qquad b^\dagger|2n+1\rangle = \sqrt{2n+2}|2n+2\rangle$$

where p is the order of paraquantization $p = 2h_0$, $h_0 = \frac{1}{2}, 1, \frac{3}{2}, \ldots$ [4, 5]. Here the eigenvalues of the operator $h(q)$ are easily deduced in the forms

$$\alpha_{2n+1} = 2n + p + g(q)2n \qquad \alpha_{2n+2} = 2n + 2 + g(q)(2n+p). \tag{17}$$

The corresponding parabosonic deformation is thus characterized by

$$bb^\dagger + g(q)b^\dagger b = h(q) \tag{18a}$$

with

$$h(q)|2n\rangle = \alpha_{2n+1}|2n\rangle \qquad h(q)|2n+1\rangle = \alpha_{2n+2}|2n+1\rangle \tag{18b}$$

which does not seem included in the framework (1). In fact we need the information on q-parabosons in order to be satisfied also in the present case.

(iii) *For q-parabosons*, a first attempt has recently been suggested [6] by simply replacing the above eigenvalues (17) by their brackets, so that we have

$$\alpha'_{2n+1} = [2n+p] + g(q)[2n] \qquad \alpha'_{2n+2} = [2n+2] + g(q)[2n+p] \tag{19}$$

with $[x]$ defined according to (11). Such a deformation continues to ask for differences between even and odd cases. So, let us eliminate this point by requiring different definitions of the brackets for even and odd numbers but for the whole set of states. According to that new proposition and to the relations (16), we ask for

$$\begin{aligned}
b|2n\rangle &= \sqrt{|[2n]|}\,|2n-1\rangle \xrightarrow[q\to1]{} \sqrt{2n}\,|2n-1\rangle \\
b|2n+1\rangle &= \sqrt{|[2n+1]|}\,|2n\rangle \xrightarrow[q\to1]{} \sqrt{2n+p}\,|2n\rangle \\
b^\dagger|2n\rangle &= \sqrt{|[2n+1]|}\,|2n+1\rangle \xrightarrow[q\to1]{} \sqrt{2n+p}\,|2n+1\rangle \\
b^\dagger|2n+1\rangle &= \sqrt{|[2n+2]|}\,|2n+2\rangle \xrightarrow[q\to1]{} \sqrt{2n+2}\,|2n+2\rangle.
\end{aligned} \tag{20}$$

We thus need

$$|[2n]| \xrightarrow[q\to1]{} 2n \qquad \text{and} \qquad |[2n+1]| \xrightarrow[q\to1]{} 2n+p \tag{21}$$

and define the usual bracket (11) in the first case while

$$[2n+1]=\frac{q^{1/2}}{1-q}(q^{-n-p/2}-q^{n+p/2}). \tag{22}$$

For $p=2$, for example, we then obtain the eigenvalues in the forms:

$$\alpha_{2n+1}=|[2n+1]|+|[2n]|g(q) \tag{23}$$

and

$$\alpha_{2n+2}=|[2n+2]|+|[2n+1]|g(q)$$

leading to the same relation obtained for q-bosons (see (13)) and to the q-parabosonic deformation (14) *but* with the definitions (11) *and* (22). Once again now the set (1)-(3) is convenient for these q-parabosons. Moreover, by taking the limit $q\to 1$, we also get the previous parabosonic case while, when $p=1$, we identify q-parabosons with q-bosons.

The second step now consists to analyse the *Fermi-like context* and to consider successively (iv) q-fermions, (v) parafermions and (vi) q-parafermions.

(iv) *For q-fermions*, the coefficients and the bracket can be chosen [1] as follows with $n=0,1,\ldots,\beta$ $(\geqslant 1)$:

$$|c_n|^2=\frac{1}{|[n]!|} \qquad [n]=\frac{q^{1/2}}{1+q}(q^{-n/2}-(-1)^n q^{n/2}). \tag{24}$$

Consequently we once again obtain the welcome results (12)-(14). If the function $g(q)$ is now $q^{1/2}$, we immediately get the typical anticommutation relation

$$ff^\dagger+q^{1/2}f^\dagger f=q^{-N/2} \tag{25}$$

reducing to the expected anticommutator when $q\to 1$ in the purely fermionic case.

(v) *For parafermions* [5], we are limited by a finite number of states $|n\rangle$ through the values $n=0,\ldots,p$ where p is once again the order of paraquantization. Due to the property [5]

$$f^\dagger|n\rangle=\sqrt{(n+1)(p-n)}\,|n+1\rangle \tag{26a}$$

or evidently

$$(f^\dagger)^n|0\rangle=\sqrt{\frac{n!p!}{(p-n)!}}|n\rangle \tag{26b}$$

we obtain from (5) and (26b) that

$$|c_n|^2=\frac{(p-n)!}{n!p!} \tag{27}$$

so that (10) leads to

$$\frac{n!p!}{(p-n)!}=\prod_{k=1}^{n}\left[\sum_{j=0}^{k-1}(-1)^j g^j(q)\alpha_{k-j}\right]. \tag{28}$$

The eigenvalues of $h(q)$ then appears for $l=1,2,\ldots,p+1$ as given by

$$\alpha_l=l(p-l+1)+(l-1)(p-l+2)g(q) \tag{29}$$

and we finally get a paradeformation characterized by

$$ff^{\dagger}+g(q)f^{\dagger}f=(N+1)(p-N)+g(q)N(p-N+1) \tag{30}$$

where $g(q)$ stands for an arbitrary function. If $g(q)$ is unity we recover the Ohnuki-Kamefuchi relation [5]. At this point, the relation (30) does not seem to belong to our scheme except if we take care of the next context when $q\to 1$.

(vi) *For q-parafermions*, the simplest idea [6] is to replace the numbers in (27) by brackets limited by $n=0,1,\ldots,p$ due to factors like $p-n$. Here again we ask for another choice, i.e.

$$|c_n|^2=\frac{1}{|[n]!|} \qquad n=0,1,\ldots,\beta \tag{31}$$

with $\beta\geqslant p$. This new proposal includes the q-fermions (corresponding to $p=1$) and evidently leads to (1)-(3) with in particular

$$ff^{\dagger}+g(q)f^{\dagger}f=|[N+1]|+g(q)|[N]| \tag{32}$$

but with *ad hoc* definitions of brackets for every value of p. Through the parafermionic properties (26a) and the relations (2) and (3), we are asking that

$$\begin{aligned}
f^{\dagger}|0\rangle &= \sqrt{|[1]|}\,|1\rangle \xrightarrow[q\to 1]{} \sqrt{p}\,|1\rangle \\
f^{\dagger}|1\rangle &= \sqrt{|[2]|}\,|2\rangle \xrightarrow[q\to 1]{} \sqrt{2(p-1)}\,|2\rangle \\
&\vdots \\
f^{\dagger}|p\rangle &= \sqrt{|[p+1]|}\,|p+1\rangle \xrightarrow[q\to 1]{} 0.
\end{aligned} \tag{33}$$

We thus require for arbitrary p:

$$[0]=0 \qquad |[1]|=p \qquad |[2]|=2(p-1),\ldots \tag{34a}$$

and

$$|[p+1]|\to 0 \qquad \text{when } q\to 1. \tag{34b}$$

If we apply such considerations for fixed $p=2$ as an example, we want

$$[0]=0 \qquad |[1]|=2 \qquad |[2]|=2 \qquad |[3]|\to 0 \qquad \text{when } q\to 1. \tag{35}$$

This is immediately realized with even and odd brackets defined as follows

$$[2m]=2\frac{1-(-1)^m q^{2m}}{1+q^2} \qquad [2m+1]=2\frac{1-(-1)^{m+1}q^{m+1}}{1+q}. \tag{36}$$

Let us finally stress on the fact that (32) and (35) lead to a formalism which is equivalent to the one characterizing parafermionic operators (see (30)). These latest oscillators are thus also included in our proposal.

At this stage we have thus included the above six contexts in the general framework (1)-(3) which can, moreover, lead to equivalent infinite families of deformations due to the still left arbitrary functions $g(q)$.

Let us now end this letter with three brief comments concerning *first* its connection with another recent contribution [2] accommodating the various methods of quantization found in the literature, *secondly* the fact that parafermions and q-fermions can be related in some specific cases and *thirdly* the fact that we can point out the existence of q-parasuperalgebras [7, 8].

With respect to the work of Odaka *et al* [2] we have extended the discussion to *complex* values of q including in particular the roots of the unity and to the *trilinear* characteristics of the parastatistical developments. In particular, this has implied the operatorial character of $h(q)$ and correspondingly some specific definitions of different brackets in order to classify and to unify all the cases encountered in the literature.

In order to show that parafermions and q-fermions can be related, let us take the case $p=2$. From (26), we get

$$f^{\dagger}|0\rangle=\sqrt{2}\,|1\rangle \qquad f^{\dagger}|1\rangle=\sqrt{2}\,|2\rangle \qquad f^{\dagger}|2\rangle=0 \tag{37}$$

and, from (30), we have

$$ff^{\dagger}+g(q)f^{\dagger}f=(N+1)(2-N)+g(q)N(3-N). \tag{38}$$

For q-fermions, with the bracket (24), we see that

$$[0]=0 \qquad [1]=1 \qquad [2]=q^{-1/2}-q^{1/2} \qquad [3]=q+q^{-1}-1. \tag{39}$$

The brackets [2] and [3] can thus be evaluated for

$$q=\exp\left(\pm\frac{\mathrm{i}\pi}{3}\right) \qquad \text{or} \qquad q=\exp\left(\pm\frac{5\mathrm{i}\pi}{3}\right) \tag{40}$$

which are roots of the unity ($q^6=1$). We then get

$$[0]=0 \qquad [1]=1 \qquad [2]=1 \qquad [3]=0$$

these values being the necessary ones ensuring the equivalence between (3) *and* (37), for example, up to a constant factor. Let us also mention that such considerations do not work for $p=3$-parafermions.

Finally, by considering the superposition of ordinary bosons characterized by

$$b|n\rangle=\sqrt{n}\,|n-1\rangle \qquad \text{and} \qquad b^{\dagger}|n\rangle=\sqrt{n+1}\,|n+1\rangle \tag{41a}$$

with q-parafermions characterized by

$$f|m\rangle=\sqrt{|[m]|}\,|m-1\rangle \qquad \text{and} \qquad f^{\dagger}|m\rangle=\sqrt{|[m+1]|}\,|m+1\rangle \tag{41b}$$

we can construct the parasupercharge(s)

$$Q=bf^{\dagger} \qquad Q^{\dagger}=b^{\dagger}f \tag{42}$$

and show that

$$\begin{aligned}(Q^2Q^{\dagger}+QQ^{\dagger}Q+Q^{\dagger}Q^2)|n,m\rangle\\ =\{n|[m+1]|+(n+1)|[m]|+(n-1)|[m+2]|\}Q|n,m\rangle \xrightarrow[q\to 1]{} 4HQ|n,m\rangle\end{aligned} \tag{43a}$$

as well as that

$$\begin{aligned}(Q^{\dagger 2}Q+Q^{\dagger}QQ^{\dagger}+QQ^{\dagger 2})|n,m\rangle\\ =\{n|[m+1]|+(n+1)|[m]|+(n+2)|[m-1]|\}Q^{\dagger}|n,m\rangle \xrightarrow[q\to 1]{} 4HQ^{\dagger}|n,m\rangle\end{aligned} \tag{43b}$$

where H is the Rubakov–Spiridonov Hamiltonian [8] if $p=2$ and $m=0, 1, 2$. In that particular case, we thus get with the characteristics (35) that the relations (43) tend (for $q\to 1$) to the ones defining a parasuperalgebra [8]. We are just able to announce the existence of q-parasuperalgebras through equations (43). Let us also notice that, for arbitrary p, we immediately deduce from equations (41b)

$$[f,[f^\dagger,f]]=(2[N]-[N+1]-[N-1])f. \tag{44}$$

Moreover, for every value of N, we notice that, through equations (34), we have

$$[f,[f^\dagger,f]]\xrightarrow[q\to 1]{} 2f \qquad \forall p. \tag{45}$$

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